Midterm 2

EECS 245, Fall 2026 at the University of Michigan

Name: _____

uniqname: _____

UMID: _____

Room: ◯ 1013 DOW ◯ 1014 DOW ◯ 4721 BBB ◯ Other

Instructions

- The student exam consists of 7 problems, worth a total of 100 points, spread across 14 pages (7 sheets of paper).
- You have 120 minutes to complete this exam, unless you have extended-time accommodations through SSD.
- Write your uniqname in the top right corner of each page.
- For free response problems, **you must show all of your work**, and write your final answer in the space provided. We will not grade work that appears elsewhere, and you may lose points if your work is not shown.
- For multiple choice problems, completely fill in bubbles and square boxes; if we cannot tell which option(s) you selected, you may lose points.
  ◯ A bubble means that you should only select one choice.
  ☐ A square box means you should select all that apply.
- You may refer to **two double-sided 8.5x11" handwritten notes sheets**. Other than that, you may not refer to any other resources or technology during the exam (no phones, watches, or calculators).

You are to abide by the University of Michigan/Engineering Honor Code. To receive a grade, please sign below to signify that you have kept the Honor Code pledge.

I have neither given nor received aid on this exam, nor have I concealed any violations of the Honor Code.

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Tip: Skim through the entire exam before starting to work on it.

Problem 1 (10 pts)

Let

$$A = \begin{bmatrix} 6 & \alpha \\ 8 & 12 \end{bmatrix}$$

Each part below is independent. Find the value of α that satisfies the given condition.

a) (3 pts) $\det(A) = 40$.

b) (3 pts) $\text{rank}(A) = 1$.

c) (4 pts) $A^{-1} = \begin{bmatrix} 3/2 & -1 \\ -1 & 3/4 \end{bmatrix}$.

Problem 2 (10 pts)

Let

$$B = \begin{bmatrix} 2 & 2 & 2 \\ 3 & -3 & 0 \\ 1 & 1 & -2 \end{bmatrix}$$

a) (4 pts) Find BB^T.

b) (6 pts) Find B^{-1}. Fully simplify each entry. *Hint: Use part (a), not* $\det(B)$ *or a* 3×3 *inverse formula. If* $BB^T = C$*, what is* B^{-1}*?*

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Problem 3 (16 pts)

Let $X = CR$, where a, b are unknown scalars. Furthermore, suppose that

$$C = \begin{bmatrix} 4 & 0 \\ -2 & 5 \\ 3 & -1 \\ 0 & 6 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 3 & a & -2 & 5 \\ 0 & 4 & 8 & b & 0 \end{bmatrix}, \qquad \begin{bmatrix} -5 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \in \text{nullsp}(X)$$

a) (4 pts) Determine the following values.

$\text{rank}(X) =$ $\boxed{}$ $\qquad \dim(\text{nullsp}(X)) =$ $\boxed{}$

b) (8 pts) Find a and b.

c) (4 pts) Which of the following choices of columns of X form a basis for $\text{colsp}(X)$? **Select all that apply.**

- ☐ Columns 1 and 2
- ☐ Columns 1 and 5
- ☐ Columns 2 and 3
- ☐ Columns 3 and 4
- ☐ Columns 4 and 5
- ☐ Columns 1, 2, and 3

Problem 4 (16 pts)

Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

a) (2 pts) Find $A\vec{v}$.

b) (2 pts) What is the smallest positive integer k such that $A^k = I$?

◯ 2 ◯ 3 ◯ 4 ◯ 6 ◯ 8 ◯ 12

c) (6 pts) Give one basis for nullsp$(A - I)$.

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Recall,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

d) (6 pts) Applying A to a vector $\vec{x}$ changes its components by the amounts shown below:

$$A\vec{x} - \vec{x} = \begin{bmatrix} m \\ -3 \\ 5 \\ 2 \end{bmatrix}$$

(i) (4 pts) Find m. Your answer should be a number with no variables.

(ii) (2 pts) For this value of m, how many vectors $\vec{x}$ satisfy $A\vec{x} - \vec{x} = \begin{bmatrix} m \\ -3 \\ 5 \\ 2 \end{bmatrix}$?

◯ 0 ◯ 1 ◯ 2 ◯ 3 ◯ 4 ◯ Infinitely many

Problem 5 (16 pts)

Let

$$X = \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 2 & 4 \\ 0 & 2 \end{bmatrix}, \qquad \vec{q}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

We would like to find **orthogonal** unit vectors $\vec{q}_1, \vec{q}_2 \in \mathbb{R}^4$ such that

$$\text{span}(\{\vec{q}_1, \vec{q}_2\}) = \text{colsp}(X)$$

a) (6 pts) Find one possible $\vec{q}_2$.

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For the remaining parts, suppose Q has orthonormal columns $\vec{q}_1, \vec{q}_2 \in \mathbb{R}^4$. You do not need your answer to part (a). Let

$$Q = \begin{bmatrix} | & | \\ \vec{q}_1 & \vec{q}_2 \\ | & | \end{bmatrix}, \qquad Z = \begin{bmatrix} | & | \\ \vec{q}_1 + \vec{q}_2 & \vec{q}_1 - \vec{q}_2 \\ | & | \end{bmatrix}$$

and suppose $\vec{y} \in \mathbb{R}^4$ satisfies

$$Q^T \vec{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

b) (2 pts) Is $\vec{y} \in \text{colsp}(Q)$?

◯ Yes ◯ No ◯ Impossible to tell

Recall that Q has orthonormal columns, and

$$Z = \begin{bmatrix} | & | \\ \vec{q}_1 + \vec{q}_2 & \vec{q}_1 - \vec{q}_2 \\ | & | \end{bmatrix}, \qquad Q^T \vec{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

c) (4 pts) Which row gives both the vector $\vec{w}^*$ that minimizes $\|\vec{y} - Z\vec{w}\|^2$ and the resulting projection, $\vec{p} = Z\vec{w}^*$?

Choice	$\vec{w}^*$	$\vec{p} = Z\vec{w}^*$
◯ A	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$	$3\vec{q}_1 + \vec{q}_2$
◯ B	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$3\vec{q}_1 + \vec{q}_2$
◯ C	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$	$4\vec{q}_1 + 2\vec{q}_2$
◯ D	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$2\vec{q}_1 + \vec{q}_2$
◯ E	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$3\vec{q}_1 + \vec{q}_2$

d) (4 pts) Identify whether each statement is true or false.

(i) (2 pts) For every $\vec{x} \in \mathbb{R}^2$, $Q^T Q\vec{x} = \vec{x}$. ◯ True ◯ False

(ii) (2 pts) QQ^T is invertible. ◯ True ◯ False

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Problem 6 (14 pts)

Suppose we'd like to fit a multiple linear regression model to predict the daily electricity use of household i, y_i, in kilowatt-hours.

For each household i, the feature vector is $\vec{x}_i = \begin{bmatrix} x_i^{(1)} \\ x_i^{(2)} \end{bmatrix}$, where:

- $x_i^{(1)}$ is the number of residents.
- $x_i^{(2)}$ is 2 if the household uses a heat pump and 0 if it uses gas heating.

Our model is

$$h(\vec{x}_i) = w_0 + w_1 x_i^{(1)} + w_2 x_i^{(2)}$$

The dataset below has two unknown values, a and b.

i	Residents ($x_i^{(1)}$)	Heating source ($x_i^{(2)}$)	Electricity use (y_i)
1	2	a	12
2	5	2	30
3	2	2	16
4	4	0	b
5	3	0	16

a) (4 pts) Write the design matrix X. Your answer should be a matrix in terms of a and constants.

Recall, our model is $h(\vec{x}_i) = w_0 + w_1 x_i^{(1)} + w_2 x_i^{(2)}$, where $x_i^{(1)}$ is the number of residents and $x_i^{(2)}$ is 2 for a heat pump and 0 for gas heating.

i	Residents ($x_i^{(1)}$)	Heating source ($x_i^{(2)}$)	Electricity use (y_i)
1	2	a	12
2	5	2	30
3	2	2	16
4	4	0	b
5	3	0	16

b) (10 pts) The parameter vector that minimizes mean squared error is $\vec{w}^* = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$.

Find a and b. Your answers should be numbers with no variables.
Hint: What properties must the error vector satisfy?

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Problem 7 (18 pts)

Let $\vec{x} \in \mathbb{R}^2$, and consider

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad f(\vec{x}) = \left(\vec{x}^T A \vec{x} - 1\right)^2$$

a) (6 pts) Which of the following is the fully simplified expression for $\nabla f(\vec{x})$?
(i) (2 pts)

◯ $2(\vec{x}^T A\vec{x} - 1)A\vec{x}$ ◯ $4(\vec{x}^T A\vec{x} - 1)\vec{x}$ ◯ $4(\vec{x}^T A\vec{x} - 1)A\vec{x}$ ◯ $4(\vec{x}^T A\vec{x})A\vec{x}$

(ii) (4 pts) Justify your answer.

b) (4 pts) Identify whether each statement is true or false. *Hint: There is no need to think about second partial derivatives.*

(i) (2 pts) f is convex. ◯ True ◯ False

(ii) (2 pts) f attains its global minimum at exactly one vector. ◯ True ◯ False

Recall, $\vec{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad f(\vec{x}) = \left(\vec{x}^T A \vec{x} - 1\right)^2$$

c) (8 pts) We run gradient descent on f twice, starting from the **same** unknown vector $\vec{x}^{(0)}$.

- With step size $\alpha = \frac{1}{4}$, the first iteration gives $\vec{x}^{(1)} = \frac{1}{4}\begin{bmatrix} -14 \\ 1 \end{bmatrix}$.
- With step size $\alpha = \frac{3}{4}$, the first iteration gives $\vec{x}^{(1)} = \frac{1}{4}\begin{bmatrix} -54 \\ 11 \end{bmatrix}$.

Find $\vec{x}^{(0)}$. Your answer should be a vector with no variables. You can answer this part without using your answer to part (a).