Midterm 2 Solutions

EECS 245, Fall 2026 at the University of Michigan

Name: _____

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Room: ◯ 1013 DOW ◯ 1014 DOW ◯ 4721 BBB ◯ Other

Instructions

- The student exam consists of 7 problems, worth a total of 100 points, spread across 14 pages (7 sheets of paper).
- You have 120 minutes to complete this exam, unless you have extended-time accommodations through SSD.
- Write your uniqname in the top right corner of each page.
- For free response problems, **you must show all of your work**, and write your final answer in the space provided. We will not grade work that appears elsewhere, and you may lose points if your work is not shown.
- For multiple choice problems, completely fill in bubbles and square boxes; if we cannot tell which option(s) you selected, you may lose points.
  ◯ A bubble means that you should only select one choice.
  ☐ A square box means you should select all that apply.
- You may refer to **two double-sided 8.5x11" handwritten notes sheets**. Other than that, you may not refer to any other resources or technology during the exam (no phones, watches, or calculators).

You are to abide by the University of Michigan/Engineering Honor Code. To receive a grade, please sign below to signify that you have kept the Honor Code pledge.

I have neither given nor received aid on this exam, nor have I concealed any violations of the Honor Code.

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Tip: Skim through the entire exam before starting to work on it.

Problem 1 (10 pts)

Let

$$A = \begin{bmatrix} 6 & \alpha \\ 8 & 12 \end{bmatrix}$$

Each part below is independent. Find the value of α that satisfies the given condition.

a) (3 pts) $\det(A) = 40$.

Solution: For this 2×2 matrix,

$$\det(A) = 6(12) - 8\alpha = 72 - 8\alpha$$

Thus,

$$72 - 8\alpha = 40 \quad \Longrightarrow \quad \boxed{\alpha = 4}$$

b) (3 pts) $\text{rank}(A) = 1$.

Solution: Since A is a nonzero 2×2 matrix, it has rank 1 exactly when its determinant is 0. Therefore,

$$72 - 8\alpha = 0 \quad \Longrightarrow \quad \boxed{\alpha = 9}$$

For this value, the second column is $3/2$ times the first column.

c) (4 pts) $A^{-1} = \begin{bmatrix} 3/2 & -1 \\ -1 & 3/4 \end{bmatrix}$.

Solution: The inverse formula gives

$$A^{-1} = \frac{1}{72 - 8\alpha} \begin{bmatrix} 12 & -\alpha \\ -8 & 6 \end{bmatrix}$$

Matching the top-left entry gives

$$\frac{12}{72 - 8\alpha} = \frac{3}{2} \quad \Longrightarrow \quad 72 - 8\alpha = 8 \quad \Longrightarrow \quad \boxed{\alpha = 8}$$

Substituting $\alpha = 8$ gives all four entries of the specified inverse.

Problem 2 (10 pts)

Let

$$B = \begin{bmatrix} 2 & 2 & 2 \\ 3 & -3 & 0 \\ 1 & 1 & -2 \end{bmatrix}$$

a) (4 pts) Find BB^T.

Solution: The entries of BB^T are the dot products of the rows of B. The three rows are mutually orthogonal, and their squared lengths are

$$2^2 + 2^2 + 2^2 = 12, \qquad 3^2 + (-3)^2 = 18, \qquad 1^2 + 1^2 + (-2)^2 = 6$$

Hence,

$$\boxed{BB^T = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 6 \end{bmatrix}}$$

b) (6 pts) Find B^{-1}. Fully simplify each entry. *Hint: Use part (a), not* $\det(B)$ *or a* 3×3 *inverse formula. If* $BB^T = C$*, what is* B^{-1}*?*

Solution: Let $C = BB^T$. Since C is invertible,

$$B(B^T C^{-1}) = (BB^T)C^{-1} = CC^{-1} = I$$

Thus, $B^{-1} = B^T C^{-1}$. Using part (a),

$$B^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1/12 & 0 & 0 \\ 0 & 1/18 & 0 \\ 0 & 0 & 1/6 \end{bmatrix} = \boxed{\begin{bmatrix} 1/6 & 1/6 & 1/6 \\ 1/6 & -1/6 & 1/6 \\ 1/6 & 0 & -1/3 \end{bmatrix}}$$

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Problem 3 (16 pts)

Let $X = CR$, where a, b are unknown scalars. Furthermore, suppose that

$$C = \begin{bmatrix} 4 & 0 \\ -2 & 5 \\ 3 & -1 \\ 0 & 6 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 3 & a & -2 & 5 \\ 0 & 4 & 8 & b & 0 \end{bmatrix}, \qquad \begin{bmatrix} -5 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \in \text{nullsp}(X)$$

a) (4 pts) Determine the following values.

$\text{rank}(X) =$ | 2 | $\qquad \dim(\text{nullsp}(X)) =$ | 3 |

Solution: The columns of C are independent, and the rows of R are independent. This CR factorization therefore has $\text{rank}(X) = 2$. Since X has five columns, rank-nullity gives $\dim(\text{nullsp}(X)) = 5 - 2 = 3$.

b) (8 pts) Find a and b.

Solution: Let $\vec{v}$ be the given vector in $\text{nullsp}(X)$. Then $CR\vec{v} = \vec{0}$. Since the columns of C are linearly independent, this requires $R\vec{v} = \vec{0}$. Therefore,

$$R\vec{v} = \begin{bmatrix} -5 + a + 4 + 5 \\ 8 - 2b \end{bmatrix} = \begin{bmatrix} a + 4 \\ 8 - 2b \end{bmatrix} = \vec{0}$$

It follows that $\boxed{a = -4,\ b = 4}$.

c) (4 pts) Which of the following choices of columns of X form a basis for $\text{colsp}(X)$? **Select all that apply.**

- ■ Columns 1 and 2
- ☐ Columns 1 and 5
- ■ Columns 2 and 3
- ☐ Columns 3 and 4
- ■ Columns 4 and 5
- ☐ Columns 1, 2, and 3

Solution: Because C has linearly independent columns, we can check independence using the corresponding columns of R, with $a = -4$ and $b = 4$. Columns 1 and 5 are scalar multiples, as are columns 3 and 4. Each remaining listed pair is independent. Since $\text{rank}(X) = 2$, the valid bases are **columns 1 and 2, columns 2 and 3, and columns 4 and 5**. Three columns cannot form a basis for this two-dimensional space.

Problem 4 (16 pts)

Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

a) (2 pts) Find $A\vec{v}$.

Solution: Multiplying by A cycles the first three components and negates the fourth:

$$A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_3 \\ v_1 \\ v_2 \\ -v_4 \end{bmatrix} \implies \boxed{A\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -4 \end{bmatrix}}$$

b) (2 pts) What is the smallest positive integer k such that $A^k = I$?

○ 2 ○ 3 ○ 4 ● 6 ○ 8 ○ 12

Solution: The first three components return to their original positions after a multiple of 3 applications of A. The fourth returns to its original sign after an even number of applications. The smallest positive integer divisible by both 3 and 2 is $\boxed{k = 6}$.

c) (6 pts) Give one basis for nullsp$(A - I)$.

Solution: A vector is in nullsp$(A - I)$ exactly when $A\vec{x} = \vec{x}$. Componentwise,

$$x_3 = x_1, \qquad x_1 = x_2, \qquad x_2 = x_3, \qquad -x_4 = x_4$$

Thus, $x_1 = x_2 = x_3$ and $x_4 = 0$. Every vector in the null space is a multiple of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, so one basis is

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

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Recall,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

d) (6 pts) Applying A to a vector $\vec{x}$ changes its components by the amounts shown below:

$$A\vec{x} - \vec{x} = \begin{bmatrix} m \\ -3 \\ 5 \\ 2 \end{bmatrix}$$

(i) (4 pts) Find m. Your answer should be a number with no variables.

(ii) (2 pts) For this value of m, how many vectors $\vec{x}$ satisfy $A\vec{x} - \vec{x} = \begin{bmatrix} m \\ -3 \\ 5 \\ 2 \end{bmatrix}$?

◯ 0 ◯ 1 ◯ 2 ◯ 3 ◯ 4 ● Infinitely many

Solution: **(i)** Applying A rearranges the first three components, so their sum does not change. Their changes must therefore sum to zero:

$$m + (-3) + 5 = 0 \quad \Longrightarrow \quad \boxed{m = -2}$$

(ii) There are **infinitely many** solutions. One particular solution is $\begin{bmatrix} 0 \\ 3 \\ -2 \\ -1 \end{bmatrix}$. Adding any vector in nullsp$(A - I)$ preserves the product with $A - I$, giving

$$\vec{x} = \begin{bmatrix} 0 \\ 3 \\ -2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \qquad t \in \mathbb{R}$$

Problem 5 (16 pts)

Let

$$X = \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 2 & 4 \\ 0 & 2 \end{bmatrix}, \qquad \vec{q}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

We would like to find **orthogonal** unit vectors $\vec{q}_1, \vec{q}_2 \in \mathbb{R}^4$ such that

$$\text{span}(\{\vec{q}_1, \vec{q}_2\}) = \text{colsp}(X)$$

a) (6 pts) Find one possible $\vec{q}_2$.

Solution: Let $\vec{x}_2$ be the second column of X. Remove its component in the direction of $\vec{q}_1$:

$$\vec{q}_1 \cdot \vec{x}_2 = \frac{4+6+8}{3} = 6$$

$$\vec{u}_2 = \vec{x}_2 - 6\vec{q}_1 = \begin{bmatrix} 4 \\ 3 \\ 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

Since $\|\vec{u}_2\| = \sqrt{4+1+4} = 3$, one answer is

$$\boxed{\vec{q}_2 = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}}$$

This is a unit vector orthogonal to $\vec{q}_1$. The first column of X is $3\vec{q}_1$, and its second column is $6\vec{q}_1 + 3\vec{q}_2$, so the two unit vectors span $\text{colsp}(X)$. The negative of this $\vec{q}_2$ is also valid.

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For the remaining parts, suppose Q has orthonormal columns $\vec{q}_1, \vec{q}_2 \in \mathbb{R}^4$. You do not need your answer to part (a). Let

$$Q = \begin{bmatrix} | & | \\ \vec{q}_1 & \vec{q}_2 \\ | & | \end{bmatrix}, \qquad Z = \begin{bmatrix} | & | \\ \vec{q}_1 + \vec{q}_2 & \vec{q}_1 - \vec{q}_2 \\ | & | \end{bmatrix}$$

and suppose $\vec{y} \in \mathbb{R}^4$ satisfies

$$Q^T \vec{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

b) (2 pts) Is $\vec{y} \in \text{colsp}(Q)$?

◯ Yes ◯ No ● Impossible to tell

Solution: **Impossible to tell.** The vector $\vec{y}_0 = 3\vec{q}_1 + \vec{q}_2$ is in $\text{colsp}(Q)$ and has the specified product with Q^T.
There are also nonzero vectors $\vec{r}$ orthogonal to $\text{colsp}(Q)$, since this is a two-dimensional subspace of $\mathbb{R}^4$. Then $\vec{y}_0 + \vec{r}$ is outside $\text{colsp}(Q)$ but has the same product with Q^T, because $Q^T\vec{r} = \vec{0}$. Both possibilities are consistent with the information given.

Recall that Q has orthonormal columns, and

$$Z = \begin{bmatrix} | & | \\ \vec{q}_1 + \vec{q}_2 & \vec{q}_1 - \vec{q}_2 \\ | & | \end{bmatrix}, \qquad Q^T \vec{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

c) (4 pts) Which row gives both the vector $\vec{w}^*$ that minimizes $\|\vec{y} - Z\vec{w}\|^2$ and the resulting projection, $\vec{p} = Z\vec{w}^*$?

Choice	$\vec{w}^*$	$\vec{p} = Z\vec{w}^*$
○ A	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$	$3\vec{q}_1 + \vec{q}_2$
● B	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$3\vec{q}_1 + \vec{q}_2$
○ C	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$	$4\vec{q}_1 + 2\vec{q}_2$
○ D	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$2\vec{q}_1 + \vec{q}_2$
○ E	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$3\vec{q}_1 + \vec{q}_2$

Solution: **B.** Adding and subtracting the columns of Z recovers $2\vec{q}_1$ and $2\vec{q}_2$, so Z and Q have the same column space. Hence,

$$\boxed{\vec{p} = QQ^T\vec{y} = 3\vec{q}_1 + \vec{q}_2}$$

But $Z\vec{w} = (w_1 + w_2)\vec{q}_1 + (w_1 - w_2)\vec{q}_2$. Independence of $\vec{q}_1, \vec{q}_2$ therefore requires

$$w_1^* + w_2^* = 3, \qquad w_1^* - w_2^* = 1 \quad \Longrightarrow \quad \boxed{\vec{w}^* = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

$Q^T\vec{y}$ gives coefficients in Q's columns, while $\vec{w}^*$ gives coefficients in Z's columns.

d) (4 pts) Identify whether each statement is true or false.

(i) (2 pts) For every $\vec{x} \in \mathbb{R}^2$, $Q^TQ\vec{x} = \vec{x}$. ● True ○ False

(ii) (2 pts) QQ^T is invertible. ○ True ● False

Solution: **(i) True.** Orthonormal columns give $Q^TQ = I_2$, so $Q^TQ\vec{x} = \vec{x}$ for every $\vec{x} \in \mathbb{R}^2$.
(ii) False. QQ^T is 4×4, and its columns lie in the two-dimensional space colsp(Q). Its rank is at most 2, so it is not invertible.

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Problem 6 (14 pts)

Suppose we'd like to fit a multiple linear regression model to predict the daily electricity use of household i, y_i, in kilowatt-hours.

For each household i, the feature vector is $\vec{x}_i = \begin{bmatrix} x_i^{(1)} \\ x_i^{(2)} \end{bmatrix}$, where:

- $x_i^{(1)}$ is the number of residents.
- $x_i^{(2)}$ is 2 if the household uses a heat pump and 0 if it uses gas heating.

Our model is

$$h(\vec{x}_i) = w_0 + w_1 x_i^{(1)} + w_2 x_i^{(2)}$$

The dataset below has two unknown values, a and b.

i	Residents ($x_i^{(1)}$)	Heating source ($x_i^{(2)}$)	Electricity use (y_i)
1	2	a	12
2	5	2	30
3	2	2	16
4	4	0	b
5	3	0	16

a) (4 pts) Write the design matrix X. Your answer should be a matrix in terms of a and constants.

Solution: The intercept contributes a column of ones, followed by the residents and heating-source features:

$$X = \begin{bmatrix} 1 & 2 & a \\ 1 & 5 & 2 \\ 1 & 2 & 2 \\ 1 & 4 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

Recall, our model is $h(\vec{x}_i) = w_0 + w_1 x_i^{(1)} + w_2 x_i^{(2)}$, where $x_i^{(1)}$ is the number of residents and $x_i^{(2)}$ is 2 for a heat pump and 0 for gas heating.

i	Residents ($x_i^{(1)}$)	Heating source ($x_i^{(2)}$)	Electricity use (y_i)
1	2	a	12
2	5	2	30
3	2	2	16
4	4	0	b
5	3	0	16

b) (10 pts) The parameter vector that minimizes mean squared error is $\vec{w}^* = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$.

Find a and b. Your answers should be numbers with no variables.
Hint: What properties must the error vector satisfy?

Solution: The predictions and errors are

$$X\vec{w}^* = \begin{bmatrix} 11+3a \\ 29 \\ 17 \\ 19 \\ 15 \end{bmatrix}, \qquad \vec{e} = \vec{y} - X\vec{w}^* = \begin{bmatrix} 1-3a \\ 1 \\ -1 \\ b-19 \\ 1 \end{bmatrix}$$

At a least-squares solution, $\vec{e}$ is orthogonal to every column of X. Orthogonality to the intercept column gives

$$(1-3a) + 1 - 1 + (b-19) + 1 = 0 \quad \Longrightarrow \quad b - 17 - 3a = 0$$

Orthogonality to the residents column gives

$$2(1-3a) + 5 - 2 + 4(b-19) + 3 = 0 \quad \Longrightarrow \quad 4b - 68 - 6a = 0$$

Substituting $b = 17 + 3a$ into the second equation gives $6a = 0$. Thus,

$$\boxed{a = 0, \qquad b = 17}$$

These values give $\vec{e} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -2 \\ 1 \end{bmatrix}$, which is also orthogonal to the heating-source column: $2(1) + 2(-1) = 0$. All the normal equations are satisfied.

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Problem 7 (18 pts)

Let $\vec{x} \in \mathbb{R}^2$, and consider

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad f(\vec{x}) = \left(\vec{x}^T A \vec{x} - 1\right)^2$$

a) (6 pts) Which of the following is the fully simplified expression for $\nabla f(\vec{x})$?
(i) (2 pts)

◯ $2(\vec{x}^T A\vec{x} - 1)A\vec{x}$ ◯ $4(\vec{x}^T A\vec{x} - 1)\vec{x}$ ● $4(\vec{x}^T A\vec{x} - 1)A\vec{x}$ ◯ $4(\vec{x}^T A\vec{x})A\vec{x}$

(ii) (4 pts) Justify your answer.

Solution: The **third expression** is correct. Let $g(\vec{x}) = \vec{x}^T A\vec{x} - 1$, so $f(\vec{x}) = g(\vec{x})^2$. Since A is symmetric,

$$\nabla g(\vec{x}) = (A + A^T)\vec{x} = 2A\vec{x}$$

The chain rule gives

$$\boxed{\nabla f(\vec{x}) = 2g(\vec{x})\nabla g(\vec{x}) = 4(\vec{x}^T A\vec{x} - 1)A\vec{x}}$$

b) (4 pts) Identify whether each statement is true or false. *Hint: There is no need to think about second partial derivatives.*

(i) (2 pts) f is convex. ◯ True ● False

(ii) (2 pts) f attains its global minimum at exactly one vector. ◯ True ● False

Solution: Both statements are **False**. Consider

$$\vec{u} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \end{bmatrix}$$

Both satisfy $\vec{x}^T A\vec{x} = 1$, so $f(\vec{u}) = f(\vec{v}) = 0$.
(i) Their midpoint is $\vec{0}$, but

$$f\left(\frac{\vec{u} + \vec{v}}{2}\right) = f(\vec{0}) = 1 > \frac{f(\vec{u}) + f(\vec{v})}{2} = 0$$

This violates the defining inequality for a convex function.
(ii) Since f is a square, it is always nonnegative. Both $\vec{u}$ and $\vec{v}$ attain 0, so the global minimizer is not unique.

Recall, $\vec{x} \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad f(\vec{x}) = \left(\vec{x}^T A \vec{x} - 1\right)^2$$

c) (8 pts) We run gradient descent on f twice, starting from the **same** unknown vector $\vec{x}^{(0)}$.

- With step size $\alpha = \frac{1}{4}$, the first iteration gives $\vec{x}^{(1)} = \frac{1}{4}\begin{bmatrix} -14 \\ 1 \end{bmatrix}$.
- With step size $\alpha = \frac{3}{4}$, the first iteration gives $\vec{x}^{(1)} = \frac{1}{4}\begin{bmatrix} -54 \\ 11 \end{bmatrix}$.

Find $\vec{x}^{(0)}$. Your answer should be a vector with no variables. You can answer this part without using your answer to part (a).

Solution: Let $\vec{g} = \nabla f(\vec{x}^{(0)})$. Both runs have the same gradient at their starting point. The update rule gives

$$\vec{x}^{(0)} - \frac{1}{4}\vec{g} = \frac{1}{4}\begin{bmatrix} -14 \\ 1 \end{bmatrix}, \qquad \vec{x}^{(0)} - \frac{3}{4}\vec{g} = \frac{1}{4}\begin{bmatrix} -54 \\ 11 \end{bmatrix}$$

Multiply the first equation by 3 and subtract the second to eliminate $\vec{g}$:

$$2\vec{x}^{(0)} = \frac{1}{4}\left(3\begin{bmatrix} -14 \\ 1 \end{bmatrix} - \begin{bmatrix} -54 \\ 11 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Therefore,

$$\boxed{\vec{x}^{(0)} = \begin{bmatrix} 3/2 \\ -1 \end{bmatrix}}$$